Journal of Applied Mathematics and Mechanics 74 (2010) 171–180

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Investigation of the effect of seepage or evaporation from a free surface by the method of circular polygons☆

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ARTICLE INFO

Article history:
Received 15 June 2009

ABSTRACT

It is proposed to use a technique developed for polygons in polar nets to integrate equations of the Fuchs class that arise when solving a wide range of problems of plane steady seepage flow using the Polubarinova-Kochina method, based on the use of the analytical theory of linear differential equations. It is shown that, for a large class of pentagons in domains where the flows,which are very characteristic of seepage problems when there is infiltration or evaporation from the free surface, have a complex velocity, the solution of the problem of determining the unknown parameters which appear in the conformal mapping can be completed. In this case, the mapping is carried out in closed form in terms of elementary functions and it is simple and convenient for subsequent application. The results obtained are used to solve the problem of seepage from a channel, taking account of the capillarity of the ground when there is evaporation from the free surface. The results of numerical calculations are presented and a hydrodynamic analysis of the effect of the basic physical parameters of the model on the dimensions of the saturation zone is given.

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It is well known [1–3] that the use of the Polubarinova-Kochina method, which is based on the application of the analytical theory of linear differential equations of the Fuchs class, in solving seepage problems in more complex formulations and with allowance for additional factors gives rise to problems of a fundamental nature. The point is that the mapping problem in the case of circular polygons can still not be fully solved solely by integration of equations of the Fuchs class. In order to verify this, it is sufficient to recall the problem of the parameters of the Christoffel–Schwarz integral for the case of rectilinear polygons. Even when the analytical apparatus for representing of the mapping function is known, its construction is quite laborious. The case, when arcs of circles are found among the sides of a polygon is far more complex and, although the form of the differential equation which the mapping function satisfies (the Schwarz equation) is also known here, this equation, unlike in the case of rectilinear polygons, not only cannot be integrated in the general case but its integration for each specific polygon involves major mathematical difficulties.

The reason for this lies in the fact that the coefficients of the differential equations (which, in the final analysis, determine the character of its integrals), in addition to the undetermined affixes of the vertices of the polygon, also contain so-called accessor parameters which are also unknown in advance [4,5]. These parameters are not completely determined by the position of the singular points of the equation and the exponents in them. It is only known that they must be uniquely defined by the geometry of the polygon but the relation between the above mentioned parameters and the geometric characteristics of a given polygon are also unknown in advance and, so far, no general and convenient method for solving this problem exists.

For the above mentioned reasons, the nature of the solutions of equations of the Fuchs class has not been investigated in depth and, in order to study their properties, it is necessary to resort to various indirect methods: the use of a Riemann boundary value problem, and the methods of Lappo-Danilevskii and Fock (see for example, Refs. 6–8). However, these general investigations have still not led to a practical solution of the new range of problems in the theory of seepage which seriously restricts the use of the Polubarinova-Kochina method. The study of special cases, which are characteristic in the case of problems in seepage, theory but, nevertheless, are of a fairly general character, is therefore of interest. Here, as has been pointed out in Refs. 9–11, circular polygons in polar nets (Ref. 5, p. 27, 135), which are bounded by arcs of concentric circles and line segments passing through the origin of the coordinates, as well as polygons which do not belong to this class but are similar to them are very typical.

☆ *Prikl. Mat. Mekh*. Vol. 74, No. 2, pp. 239–251, 2010.
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doi:10.1016/j.jappmathmech.2010.05.007

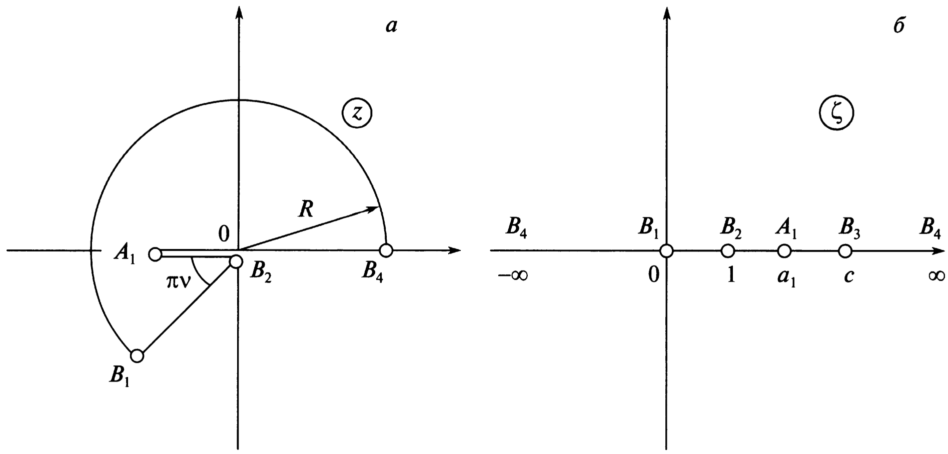

Fig. 1.

According to the generally accepted method (Ref. 5, p. 175), these polygons are transformed by means of the function $w = \ln z$ into rectilinear polygons with subsequent application of the Christoffel–Schwarz formula. However, this route leads to a large increase in the number of unknown conformal mapping parameters.

In the first place, the location of the branch points, $z = 0$ and $z = \infty$, of the logarithm, at which the conformability of the mapping is violated, with respect to the mapped domain (on the boundary or within it) has to be taken into account each time. In order to achieve uniqueness in the first case, it is necessary to draw an additional cut from the point $z = 0$ to the boundary of the domain which immediately increases the number of conformal mapping parameters by three. If, however, the point $z = 0$ lies on the boundary of the mapped domain, then an additional vertex at infinity with a zero angle in the w plane will correspond to this boundary point in the case of the above mentioned mapping, so that the number of parameters increases here also. In any case, the use of a logarithmic function in the solution will therefore lead from the very beginning to an increase in the number of parameters. This last fact, when account is taken of the addition of the corresponding coefficients in the integrand of the Christoffel–Schwarz formula (Ref. 5, p. 177), leads to very unwieldy calculations and, as a consequence, to integrals which are not expressed in terms of special functions.

Secondly, the overall number of conformal mapping parameters increases significantly when account has to be taken of the parameters which appear when solving a specific seepage problem. A particular problem is the scale constant of the modelling (see below the parameter A in formulae (4.4)) as well as the affixes of the removable singular points which occur in the physical plane but are absent in the z plane. Hence, the solution of the problem is further complicated.

Unlike the traditional approach in previous papers, [12–14] it is proposed to make use of special methods, precisely adapted to the class of polygons in polar nets considered, based on the idea of finding particular solutions of equations of the Fuchs class in the form of linear combinations with undetermined coefficients from the known particular solutions of several simpler equations with three singular points. The technique developed for polygons in polar nets is used below to solve the problem of the conformal mapping of circular pentagons which do not belong to this class, that is, polygons with a more complex structure.

The problem of the conformal mapping of a circular quadrangle in polar nets $\{\pi/2, \pi\nu, 2\pi, \pi/2\}$ with an arbitrary angle $\pi\nu$ and a cut is initially solved. Then, using the technique of "smoothing out" or "unrolling" of the angles in the beginning, [1] we consider a circular pentagon $\{\pi/2, \pi\alpha, 2\pi, \pi\beta, \pi/2\}$ with two angles, $\pi\alpha$ and $\pi\beta$, and a cut which also belongs to the class of polygons in polar nets. (Henceforth, we will use the well-known terminology and notation in Ref. 5.) Finally, using the solution obtained, the integrals of an equation of the Fuchs type, which also has five singular points but which corresponds to a circular pentagon of an even more complex structure which does not belong to the class of pentagons in polar nets, are found. Here, at all stages of the solution, the conformal mapping parameters are determined when constructing the particular solutions and, therefore, the problems of accessor parameters do not subsequently arise. Taking account of the specific features which are characteristic of the classes of polygons considered, the solutions obtained at each stage are expressed in closed form in terms of elementary functions and are therefore the simplest and most convenient for their subsequent application. Finally, the results obtained are used to solve the problem of seepage from a channel, taking account of the capillarity of the ground when there is evaporation from the free surface. The results of numerical calculations are presented and a detailed hydrodynamic analysis of the effect of the basic governing physical parameters of the model on the dimensions of the saturation zone is given.

1. Solution of the problem of the conformal mapping of a circular quadrangle $\{\pi/2, \pi\nu, 2\pi, \pi/2\}$ in polar nets. The parameter problem

Suppose a circular quadrangle $\{\pi/2, \pi\nu, 2\pi, \pi/2\}$ with an arbitrary angle $\pi\nu$ ($0<\nu<1$, $\nu=\alpha/\beta$) at the vertex B_2 and a cut with a vertex at the point A_1 is given in the z plane (Fig. 1, a). The differential quation of the Fuchs class corresponding to the problem of the conformal mapping of the ζ upper half-plane (Fig. 1, b) onto the domain considered has the form

$$\upsilon''+\left[\frac{1}{2\zeta}+\frac{1-\nu}{\zeta-1}-\frac{1}{\zeta-a_1}\right]\upsilon'+\frac{\nu(1+\nu)\zeta/4+\lambda_0}{\zeta(\zeta-1)(\zeta-a_1)}\upsilon=0 \tag{1.1}$$

where λ_0 is an accessor parameter. We recall that, in Eq. (1.1), the pre-image a_1 ($1<a_1<\infty$) of the vertex of the cut A_1, as well as the accessor parameter λ_0, remain unknown when formulating the problem and must be determined during the course of its solution.

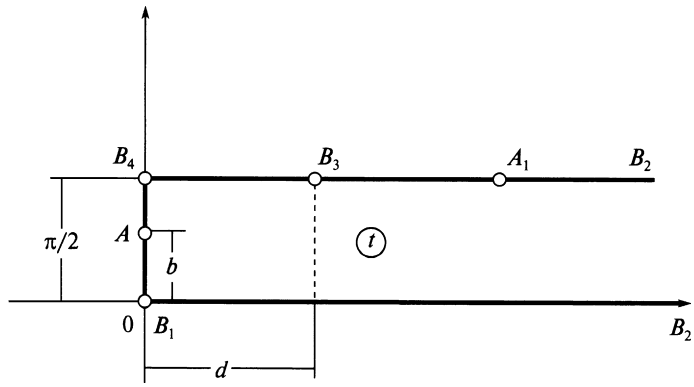

Fig. 2.

The change of variables

$$\zeta = \mathrm{th}^2 t \tag{1.2}$$

transforms Eq. (1.1) to the form

$$\begin{aligned}&[(a_1-1)\mathrm{ch}^2 t+1]\mathrm{ch}^2 t\cdot \upsilon''+[\nu(a_1-1)\mathrm{ch}^2 t+1+\nu]\mathrm{sh}^2 t\cdot \upsilon'\\&+[(\nu^2+\nu+\lambda_0)\mathrm{ch}^2 t-\nu^2-\nu]\upsilon=0\end{aligned} \tag{1.3}$$

and transfers the ζ upper half-plane into the half-strip Ret>0, 0<Imt<π/2 of the t plane (Fig. 2).

Linearly independent integrals of Eq. (1.3) have been constructed [12–14] in the form of linear combinations with undetermined coefficients of known particular solutions of a simpler hypergeometric equation [1,4,5] which, in this case, take the form

$$\upsilon_1(t)=\frac{C_1\mathrm{ch}t\,\mathrm{ch}\nu t+C_2\mathrm{sh}t\,\mathrm{sh}\nu t}{\mathrm{ch}^{1+\nu}t},\quad \upsilon_2(t)=\frac{C_1\mathrm{ch}t\,\mathrm{sh}\nu t+C_2\mathrm{sh}t\,\mathrm{ch}\nu t}{\mathrm{ch}^{1+\nu}t}, \tag{1.4}$$

where C_1 and C_2 are certain appropriate constants to be determined. If expressions (1.4) are substituted into Eq. (1.3) and the terms in ch νt and sh νt in the resulting equation are collected and equated to zero, then, in both cases, we arrive at the same homogeneous system of linear algebraic equations

$$\begin{aligned}&[\nu(1-\nu)a_1+\lambda_0]C_1+2\nu a_1 C_2=0\\&2\nu C_1+[(\nu^2-\nu-2)a_1+2+\lambda_0]C_2=0\end{aligned} \tag{1.5}$$

In order for the constants C_1 and C_2 to be non zero, it is necessary and sufficient that the determinant of system (1.5) equals zero. This requirement gives the following relation between the parameters a_1 and λ_0

$$\lambda_0^2+2\lambda_0[(\nu^2-\nu-1)a_1+1]+\nu(1+\nu)a_1[(1-\nu)(2-\nu)a_1-2]=0 \tag{1.6}$$

Note that Eq. (1.6) is identical to the Kochina condition (Ref. 1, p. 255) for the vertex of the cut A_1.

Putting $C=C_2/C_1$, we obtain expressions for the initial mapping constants:

$$C=\frac{\nu(1-\nu)a_1-\lambda_0}{2a_1\nu}=-\frac{2\nu}{(\nu^2-\nu-2)a_1+2+\lambda_0}$$

or

$$a_1=\frac{C+\nu}{C(1+\nu C)}=\frac{C(\beta+\alpha C)}{\alpha+\beta C},\quad \lambda_0=\frac{\nu(C+\nu)(1-\nu-2C)}{C(1+\nu C)}=\frac{\alpha(\beta+\alpha C)[(\beta-\alpha)C-2\beta]}{\beta^2(\alpha+\beta C)} \tag{1.7}$$

It therefore turns out that the parameter C is a uniformizing variable (Ref. 4, p. 414) for Eq. (1.6) which gives unique representations of the conformal mapping parameters a_1 and λ_0.

Equations (1.7) are the equation of a hyperbola in parametric form. If the curves (1.7) are schematically represented in the form $a=a_1(C)$ and $\lambda_0=\lambda_0(C)$, then, in the (a_1,λ_0) plane, this hyperbola passes through the point (0,0) and (0, −2) and has asymptotes

$$\lambda_0=\nu(1-\nu)a_1-2\nu^2 \quad \text{и} \quad \lambda_0=(2+\nu-\nu^2)a_1-2(1-\nu^2)$$

Thus, the functions (1.4) represent the solutions of Eq. (1.3) only for those values of the parameters a_1 and λ_0 which are the coordinates of points in the (a_1,λ_0) plane lying on the given hyperbola. Of course, in the final analysis, this last fact imposes constraints on the range of variation in the physical parameters of the hydrodynamic problem considered.

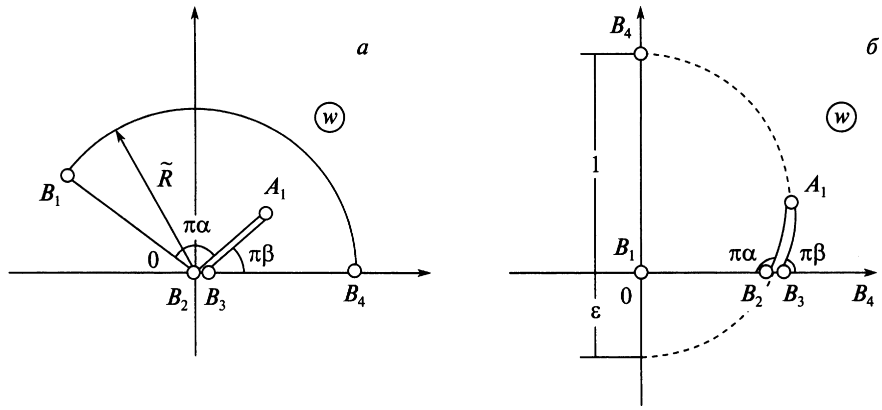

Fig. 3.

The function which completes the conformal mapping of the half-strip in the t plane onto the circular quadrangle in the z plane considered must be expressed in terms of the ratio of linear combinations of the integrals $\upsilon_1(t)$ and $\upsilon_1(t)$ (1.4). If such combinations are constructed and use is made of the correspondence of the vertices B_1, B_2 and B_4 in the t and w planes, we obtain

$$z(t) = R\exp[\pi(1+\nu)i - 2\nu t]\frac{C - \text{th}\,t}{C + \text{th}\,t} \tag{1.8}$$

The requirement of correspondence for the vertex of the cut A_1 enables us to determine the constant C which, up to the present, has remained undetermined. Note that, in the limiting case, when the cut disappears and the circular quadrangle degenerates into a triangle, we have $C = 1$, that is, $C_1 = C_2$ and, consequently, $a_1 = 1$.

The validity of expression (1.8), when relation (1.6) is satisfied, can be established by direct verification. (This refers to all the mapping functions obtained in Sections 2 - 4.)

2. Construction of the integrals of an equation of the Fuchs class with five singular points corresponding to the problem of the conformal mapping of a circular pentagon $\{\pi/2, \pi\alpha, 2\pi, \pi\beta, \pi/2\}$ in polar nets

We apply the transformation

$$w(t) = \tilde{R}\exp[\pi(\alpha+\beta)i - 2\alpha t]\left[\frac{C - \text{th}\,t}{C + \text{th}\,t}\right]^{\beta}, \quad \tilde{R} = R^{\beta} \tag{2.1}$$

to the circular quadrangle $\{\pi/2, \pi\nu, 2\pi, \pi/2\}$ in polar nets (Fig. 1, a) which transfers the initial domain into the circular pentagon $\{\pi/2, \pi\alpha, 2\pi, 2\beta, \pi/2\}$ in polar nets with angles $\pi\alpha$ and $\pi\beta$ at the vertices B_2 and B_3, which fall onto the origin of the coordinates, and a cut (Fig. 3, a). We note that

$$\pi < \arg w_{B_1} < 3\pi/2 \text{ when } 0 < \alpha/\beta < 1/2$$

$$3\pi/2 < \arg w_{B_1} < 2\pi \text{ when } 1/2 < \alpha/\beta < 1$$

The equation of the Fuchs class with five singular points, which corresponds to the problem of the conformal mapping of the upper ζ half-plane onto the resulting circular pentagon in polar nets, has the form

$$\upsilon'' + \left[\frac{1}{2\zeta} + \frac{1-\alpha}{\zeta-1} - \frac{1}{\zeta-a_1} + \frac{1-\beta}{\zeta-c}\right]\upsilon' - \frac{(\alpha+\beta)(1-\alpha-\beta)\zeta^2/4 + \lambda_1\zeta + \lambda_0}{\zeta(\zeta-1)(\zeta-a_1)(\zeta-c)}\upsilon = 0 \tag{2.2}$$

where, together with a_1, the affix $c(a_1 < c < \infty)$ corresponding to the vertex B_3 and the accessor parameters λ_1 and λ_0 still remain undetermined.

Using the Ostrogradskii–Liouville formula (Ref. 4, p. 207)

$$(\upsilon_1/\upsilon_2)'\upsilon_1^2 = \tilde{C}\exp\left[-\int_0^t p(t)dt\right], \quad \tilde{C} = \text{const}$$

where $p(t)$ is the coefficient of υ' in Eq. (2.2), from the expression for the mapping function (2.1) we find the two linearly independent integrals (2.2) in the form of the following functions of the uniformizing parameter C

$$\upsilon_1 = \upsilon(t, C), \quad \upsilon_2 = \upsilon(-t, C); \quad \upsilon(t, C) = \frac{(C + \text{th}\,t)^{\beta}\exp(\alpha t)}{(\text{ch}\,t)^{\alpha}} \tag{2.3}$$

In this case, the mapping constant a_1 takes the previous value, and the parameters $c = C^2$.

We stress that it is not so much their completeness but their symmetry which is the undoubted advantage of expressions (2.3). Other methods of solving the problem, which differ from the use of the analytical theory of linear differential equations of the Fuchs class (the use of linear-fractional transformations with subsequent application of the Christoffel–Schwarz formula), do not enable us to discern such properties of integrals (2.3).

3. Solution of the problem of the conformal mapping of a circular pentagon $\{\pi/2, \pi\alpha, 2\pi, \pi\beta, -\pi/2\}$ which does not belong to the class of polygons in polar nets. The parameter problem

We will now consider, as an example, the domain shown in Fig. 3, *b* and immediately emphasize that the circular pentagon considered, with a cut and the angles $\pi\alpha$ and $\pi\beta$ ($\beta = 1 - \alpha = 2\pi^{-1}\mathrm{arctg}\sqrt{\varepsilon}$, $0 < \varepsilon \le 1$) at the vertices B_2 and B_3, is not a polygon in polar nets, since the centre of the circle $w_0=(1-\varepsilon)i/2$ only coincides with the origin of coordinates for single value when $\alpha = \beta = 1/2$, that is, when $\varepsilon = 1$.

The equation of the Fuchs class corresponding to the problem of the conformal mapping of this circular pentagon completely retains the form of (2.2) here, and this means that the integrals (2.3) are found to be the same as before. It remains to solve the parameter problem: to relate the conformal mapping constants a_1, c, λ_1 and λ_0 to the geometric characteristics of the pentagon shown in Fig. 3, *b*.

If, following what was done in Section 1, expressions (2.3) are substituted into Eq. (2.2) which, after the change of variable (1.2), takes the form

$$[(a_1-1)(c-1)\mathrm{ch}^4 t+(a_1+c-2)\mathrm{ch}^2 t+1]\mathrm{ch}t\cdot\upsilon''+2[\alpha(a_1-1)(c-1)\mathrm{ch}^4 t$$
$$+(1+\alpha)(c-1)\mathrm{ch}^2 t+1]\mathrm{sh}t\cdot\upsilon'+[(\lambda_1+\lambda_0)\mathrm{ch}^2 t-\lambda_1]\mathrm{ch}t\cdot\upsilon=0$$

and the terms in $\mathrm{ch}^{8-i}t\,\mathrm{sh}^k t$, where $k=[1-(-1)^i]/2$, $i=0, 1, 2, \ldots, 8$ in the resulting equation are collected and equated to zero, then, in both cases, we arrive at one and the same homogeneous system of nine equations. Analysis of the equations of this system shows that the equations are identically satisfied when $i = 0, 1, 7, 8$ so that, after discarding them, the system takes a form which is completely analogous to system (1.5):

$$(C-1)^2[\alpha(\alpha-1)(a_1c-1)-\alpha(\alpha+1)(c-a_1)+\lambda_1+\lambda_0]+2(C-1)(\lambda_1+\lambda_0)$$
$$-2[(1-\alpha)(a_1-1)(c-1)-\lambda_1-\lambda_0]=0 \tag{3.1}$$

$$(C-1)^2\alpha[-\alpha(a_1-1)+c-1]+(C-1)[(\alpha^2-1)(a_1-1)\cdot(c-1)+\lambda_1+\lambda_0]$$
$$+(\alpha-1)(a_1-1)(c-1)+\lambda_1+\lambda_0=0 \tag{3.2}$$

$$(C-1)^2\{\alpha[(\alpha+1)(c-a_1)+2(\alpha-1)]-\lambda_1\}-2(C-1)\lambda_1$$
$$+2[(1-\alpha)(a_1c+2)-(2-\alpha)a_1+(2\alpha-1)c]-3\lambda_1-\lambda_0=0 \tag{3.3}$$

$$(C-1)^2\alpha(1-\alpha)+(C-1)\{(\alpha+1)[-a_1+\alpha(c-1)+1]-\lambda_1\}-a_1+\alpha(c-1)+1-\lambda_1=0 \tag{3.4}$$

$$(C-1)^2\alpha(1-\alpha)+2[a_1-\alpha(c-1)-1]+\lambda_1=0 \tag{3.5}$$

Equations (3.1), (3.2), (3.4) and (3.5) of this system enable us successively to determine all the required mapping constants in terms of the parameter C:

$$a_1=\frac{C(1-\alpha+\alpha C)}{\alpha+(1-\alpha)C},\quad c=C^2,\quad \lambda_1=\alpha(1-\alpha)(C-1)^2\frac{2-\alpha+(1+\alpha)C}{\alpha+(1-\alpha)C},$$
$$\lambda_0=\alpha(1-\alpha)(C-1)^2a_1 \tag{3.6}$$

Following the well-known procedure (Ref. 1, p. 255), we now derive the condition for the vertex of the cut A_1 in the case considered. This point must be a regular point of the solution of differential equation (2.2), that is around it the fundamental system of solutions must not contain a logarithm and, consequently, it must have the exponents (0,2). However, the derivative of these functions with respect to ζ must then also belong to the indices (0,1). Constructing the differential equation which $d\upsilon/d\zeta$ satisfies, where υ is the solution of Eq. (2.2), and using the notation $d\upsilon/d\zeta=\tau$, we obtain the following equation for τ

$$\tau''+\left[\frac{3}{2\zeta}+\frac{2-\alpha}{\zeta-1}+\frac{1+\alpha}{\zeta-c}-\frac{\lambda_1}{\lambda_1\zeta+\lambda_0}\right]\tau'+\frac{Q(\zeta)\tau}{\zeta(\zeta-1)(\zeta-a_1)(\zeta-c)(\lambda_1\zeta+\lambda_0)}=0 \tag{3.7}$$

where

$$Q(\zeta)=-(\lambda_1\zeta+\lambda_0)^2/4+(\lambda_1\zeta+\lambda_0)\{[(\zeta-a_1)(\zeta-c)+(\zeta-1)(\zeta-c)$$
$$+(\zeta-1)(\zeta-a_1)]/2+(1-\alpha)[(\zeta-a_1)(\zeta-c)+\zeta(\zeta-c)+\zeta(\zeta-a_1)]-[(\zeta-1)(\zeta-c)$$
$$+\zeta(\zeta-c)+\zeta(\zeta-1)]+\alpha[(\zeta-1)(\zeta-a_1)+\zeta(\zeta-a_1)+\zeta(\zeta-1)]\}$$
$$-\lambda_1[(\zeta-1)(\zeta-a_1)(\zeta-c)/2+(1-\alpha)\zeta(\zeta-a_1)(\zeta-c)-\zeta(\zeta-1)(\zeta-c)+\alpha\zeta(\zeta-1)(\zeta-a_1)]$$

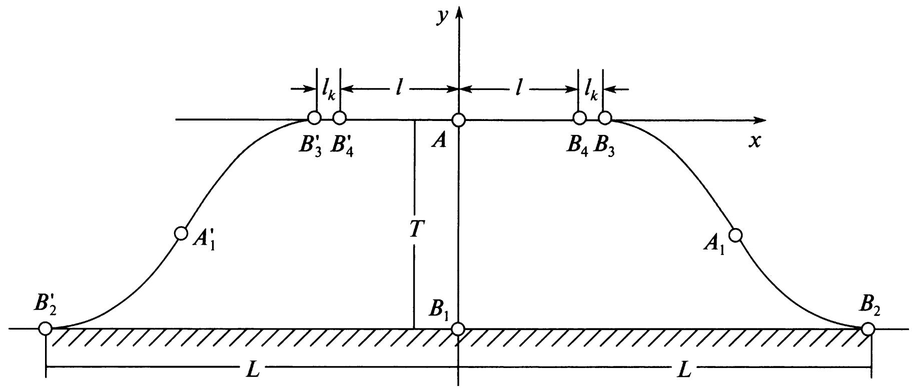

Fig. 4.

It is clear that the term with $1/(\zeta-a_1)$ vanishes in the coefficient of τ'. The condition that $\zeta=a_1$ must be a regular point of Eq. (3.7) requires that the polynomial $Q(\zeta)$ is divided by $\zeta-a_1$. This condition leads to the equation

$$(\lambda_1 a_1+\lambda_0)^2-4(\lambda_1 a_1+\lambda_0)\{[(1+2\alpha)a_1-1](c-a_1)/2-(1-\alpha)a_1(a_1-1)\}$$
$$+4\lambda_1 a_1(a_1-1)(c-a_1)=0 \tag{3.8}$$

If the values found above for the parameters of the mapping (3.6) are substituted into Eq. (3.8), we arrive at Eq. (3.3) of the system. Consequently, Kochina's condition (3.8) and Eq. (3.3) are equivalent, which enables us to eliminate Eq. (3.3) from the system.

Hence, the system of four equations (3.1), (3.2), (3.4) and (3.5) is uniquely solvable for the required parameters (3.6) as functions of the single parameter C which, as in Section 2, turns out to be the uniformizing variable (Ref. 4, p. 414) for Eq. (3.8), that is, which gives unique representations of the parameters a_1, c, λ_1 and λ_0.

Note that, in the limiting case when the cut disappears, that is, when $C=1$, Eqs (3.1), (3.2), (3.4) and (3.5) turn into an identity and, from (3.6), we obtain $a_1=c=1, \lambda_1=\lambda_0=0$.

The function completing the conformal mapping of the half- strip of the t plane (Fig. 2) onto the circular pentagon of the w plane (Fig. 3, b), which must be expressed in terms of the ratio of linear combinations of the integrals (2.3), takes the form

$$w=\sqrt{\varepsilon}\chi^-(t;\alpha,\beta)/\chi^+(t;\alpha,\beta) \tag{3.9}$$

Here,

$$\sqrt{\varepsilon}=tg\frac{\pi\beta}{2},\quad \chi^{\pm}(t;\alpha,\beta)=(C+\text{th}\,t)^{\beta}\exp(\alpha t)\pm(C-\text{th}\,t)^{\beta}\exp(-\alpha t), \tag{3.10}$$

We recall that this circular pentagon $\{\pi/2, \pi\alpha, 2\pi, \pi\beta, \pi/2\}$ does not belong to the class of polygons in polar nets and the use of linear-fractional transformations and a reduction to a rectilinear hexagon does not directly lead to formula (3.9) which clearly demonstrates the symmetry.

Note that the circular pentagon shown in Fig. 3, b is very typical of problems of underground hydromechanics and, in particular, in the study of flows with free boundaries. If uniform seepage or evaporation of an intensity ε occurs on these boundaries, the well known non-linear equation (Ref. 1, p.55)

$$u^2+\upsilon^2-(k+\varepsilon)\upsilon+k\varepsilon=0 \tag{3.11}$$

must be satisfied on the depression curve, where k is the seepage coefficient and u and υ are the components of the complex velocity $w=u+i\upsilon$. Equation (3.11) is the equation of a circle passing through the points $(0, \varepsilon)$ and $(0, k)$ with its centre at the point $(0,(k+\varepsilon)/2$ and a radius equal to $(k-\varepsilon)/2$. In the case of seepage, $\varepsilon>0$, and, when $\varepsilon<0$, we have evaporation from the free surface. Taking account of the fact that, in complex hydromechanical schemes, there is necessarily in inflection point on the depression curves (and in isolated cases there is not one), we obtain that, in the plane of the complex velocity w, the cut along the arc of the circle (3.11) represented in Fig. 3, b (when $k=1$) corresponds to the part $B_2A_1B_3$ of the free surface of the plane of the flow z. Such a configuration in domains of complex flow velocity is especially characteristic of problems involving seepage from channels.

4. Application of the results. The problem of determining the dimensions of the saturation zone in the case of seepage from a channel

We will now demonstrate the use of integrals (2.3) and the mapping function (3.9) in solving a plane steady seepage problem (using the Darcy's law).

We will consider the problem of determining the dimensions l_k and L of the saturation zone in the case of a flow from a channel with a low level of water of width $2l$ into a layer of homogeneous isotropic ground of thickness T underlain by a horizontal confining stratum (Fig. 4). The uniform evaporation from the free surface of intensity $\varepsilon(0<\varepsilon<1)$ is the discharge factor compensating the seepage from the channel.

We introduce the complex potential of the motion $w=\varphi+i\psi$, where φ is the velocity potential and ψ is the stream function, and the complex coordinate $z=x+iy$, referred to kT and T respectively. By virtue of the symmetry, we will confine ourselves to considering of the right-hand half of the flow domain $AB_1B_2A_1B_3B_4$.

We will assume that $\varphi=0$ along the channel bottom AB_4 and that $\psi=0$ along the line of symmetry AB_1. In the segment B_3B_4, we put $\psi=Q$, where Q is the required rate of seepage from the channel. Hence, according to Verigin, [15,16] the segment B_3B_4 is considered to be the impermeable line of outflow of the capillary waters onto the surface of the earth. The following boundary conditions are satisfied

$$\begin{aligned}&AB_4: y=0,\varphi=0;\quad AB_1: x=0,\psi=0\\&B_1B_2: y=-T,\psi=0;\quad B_3B_4: y=0,\psi=Q\\&B_2A_1B_3: \varphi=-y+h_k,\psi=Q-\varepsilon(x-l-l_k)\end{aligned}\tag{4.1}$$

when the coordinates one chosen as shown in Fig. 4 and the plane of comparison of the potentials and the plane $y=0$ are matched on the boundary of the domain of motion, where h_k is the static height of the capillary rise of the ground water and l_k is the required width of the capillary spreading of the water. Assuming that $x=L$ in the second condition of (4.1) for $B_2A_1B_3$, we obtain

$$Q=\varepsilon\,(L-l-l_k)\tag{4.2}$$

The last relation expresses the equality of the flow rate from the channel to the amount of evaporation from the free surface under conditions of steady seepage. The capacity of the stratum T, the channel width l and the static height of the capillary spreading of the water h_k are assumed to be given.

In this case, the domain of the complex velocity w, corresponding to boundary conditions (4.1), is the circular pentagon studied in Section 3 (Fig. 3, b) with angles at the vertices B_2 and B_3 equal to $\pi\alpha=\pi(1-\nu)$ and $\pi\beta=\pi\nu$, where ($\nu=2\operatorname{arctg}\sqrt{\varepsilon}/\pi$. In this case, the mapping function (3.9) takes the form

$$w(t)=\sqrt{\varepsilon}\chi^-(t;1-\nu,\nu)/\chi^+(t;1-\nu,\nu)\tag{4.3}$$

We use the Polubarinova-Kochina method. This is based on the use of the analytical theory of linear differential equations of the Fuchs class, [1] to solve boundary-value problem (4.1). Assuming that $w=d\omega/dz$, determining the indices of the functions $d\omega/dt$ and dz/dt close to the singular points (Ref. 1, p. 259) and taking account of relation (4.3), we arrive at the relations

$$\frac{d\omega}{dt}=\sqrt{\varepsilon}A\operatorname{ch}^{1-\nu}t\frac{\chi^-(t;1-\nu,\nu)}{\Delta(t)},\quad \frac{dz}{dt}=A\operatorname{ch}^{1-\nu}t\frac{\chi^+(t;1-\nu,\nu)}{\Delta(t)}\tag{4.4}$$

$$\Delta(t)=\sqrt{(\operatorname{sh}^2t+B^2)(\operatorname{sh}^2t+D^2)^\nu},\quad C=D/\sqrt{D^2-1},\quad B=\sin b,\quad D=\operatorname{ch}d$$

where $A>0$ is the scaling constant, and the ordinate b and the abscissa d are unknown affixes of the points A and B_3 in the t plane respectively (Fig. 2). It can be verified that the functions (4.4) satisfy conditions (4.1) which have been formulated in terms of the functions $d\omega/dt$ and dz/dt and are therefore the parametric solution of the initial boundary-value problem.

Writing representations (4.4) for different parts of the boundary of the domain t with subsequent integration over the whole contour of the auxiliary domain (Fig. 2) leads to the closure of the domain of motion z and thereby serves as a control of the calculations.

As a result, we obtain expressions for the basic geometric and seepage characteristics

$$\sqrt{\varepsilon}A\int_0^d\frac{\Phi_{B_3B_4}}{\Delta_1(t)}dt=h_k,\quad A\int_b^{\pi/2}\frac{\operatorname{Re}\Psi_{AB_4}}{\Delta_2(t)}dt=l,\quad A\sin\frac{\pi\nu}{2}\int_d^\infty\frac{\Phi_{B_2A_1B_3}}{\Delta_3(t)}dt=h_k\tag{4.5}$$

which enable us to determine the unknown parameters of the conformal mapping B and D as well as the modelling constant A. After the unknown constants have been found, the required dimensions of the saturation zone are successively calculated using the formulae

$$l_k=A\int_0^d\frac{X_{B_3B_4}}{\Delta_1(t)}dt,\quad L=A\int_0^\infty\frac{X_{B_1B_2}}{\Delta(t)}dt\tag{4.6}$$

after which the flow rate Q is determined using formula (4.2).

The other expressions for the flow rate Q and the capacity T:

$$Q=A\int_b^{\pi/2}\frac{\operatorname{Im}\Psi_{AB_4}}{\Delta_2(t)}dt=A\cos\frac{\pi\nu}{2}\int_d^\infty\frac{\Phi_{B_2A_1B_3}}{\Delta_3(t)}dt,\quad T=A\int_0^b\frac{\operatorname{Re}\Psi_{AB_1}}{\Delta_4(t)}dt\tag{4.7}$$

Table 1

Varied parameter	$L\times10^3$	$l_k\times10^4$	$Q\times10^3$	Varied parameter	$L\times10^3$	$l_k\times10^4$	$Q\times10^3$
$h_k\times10^2=10$	3461	425	888	$\varepsilon\times10^2=10$	7752	4998	605
25	3769	1207	979	25	5008	2742	884
40	4031	2163	1046	40	4031	2163	1046
55	4271	3291	1097	55	3495	1888	1158
70	4501	4585	1137	70	3147	1722	1243
$l\times10^2=30$	2326	1577	748	$T\times10^2=50$	2408	4245	313
75	3372	1943	971	125	3278	2761	721
120	4031	2163	1046	200	4031	2163	1046
165	4571	2285	1077	275	4644	1880	1302
210	5063	2349	1091	350	5128	1733	1502

serve as a control on the calculation. In expressions (4.5)–(4.7) (the upper sign on the right is chosen for the upper function on the left)

$$\left.\begin{matrix}\Phi_{B_3B_4}\\ X_{B_3B_4}\end{matrix}\right\}=(C\operatorname{sh}t+\operatorname{ch}t)^{\nu}\exp(1-\nu)t\pm(\operatorname{ch}t-C\operatorname{sh}t)^{\nu}\exp(\nu-1)t$$

$$\left.\begin{matrix}\Psi_{AB_4}\\ X_{AB_4},Y_{AB_1}\end{matrix}\right\}=(C\cos t+i\sin t)^{\nu}\exp i(1-\nu)t\mp(C\cos t-i\sin t)^{\nu}\exp i(\nu-1)t$$

$$\left.\begin{matrix}\Phi_{B_2A_1B_3}\\ \Psi_{B_2A_1B_3}\end{matrix}\right\}=(C\operatorname{sh}t+\operatorname{ch}t)^{\nu}\exp(1-\nu)t\pm(C\operatorname{sh}t-\operatorname{ch}t)^{\nu}\exp(\nu-1)t$$

$$X_{B_1B_2}=(C\operatorname{ch}t+\operatorname{sh}t)^{\nu}\exp(1-\nu)t+(C\operatorname{ch}t-\operatorname{sh}t)^{\nu}\exp(\nu-1)t$$

$$\Delta_1(t)=\sqrt{(\operatorname{ch}^2t-B^2)(D^2-\operatorname{ch}^2t)^{\nu}},\quad \Delta_2(t)=\sqrt{(\sin^2t-B^2)(D^2-\sin^2t)^{\nu}}$$

$$\Delta_3(t)=\sqrt{(\operatorname{ch}^2t-B^2)(\operatorname{ch}^2t-D^2)^{\nu}},\quad \Delta_4(t)=\sqrt{(B^2-\sin^2t)(D^2-\sin^2t)^{\nu}}$$

In the limiting case when $h_k=0$, that is, there is no ground capillarity, the vertex of the cut A_1 in the plane of the complex velocity w reaches the ordinate axis and the circular pentagon degenerates into an circular triangle: in the flow plane z, the point of inflection A_1, on merging with the point B_3, reaches the abscissa. In this case, the parameters $d=0$, $D=1$ and $C=\infty$, and it follows from the first formula of (4.6) that $l_k=0$ and the results found earlier are obtained. [17]

5. Analysis of the numerical results

The flow pattern calculated for

$$h_k=0.4,\quad \varepsilon=0.4,\quad l=1.2,\quad T=2.0 \tag{5.1}$$

is shown in Fig. 4.

The results of the calculations of the effect of the governing physical parameters h_k, ε, l and T on the dimensions of the saturation zone L, l_k and the flow rate Q are presented in Table 1 which is subdivided by double lines into four blocks. In each of them, one of the above mentioned parameters is varied in the admissible range while the values of the remaining parameters are fixed in accordance with equalities (5.1). The dependences of l_k and L on the parameters ε, h_k, l and T respectively are shown in Fig. 5.

An analysis of the data in the table and the graphs enables us to draw the following conclusions.

An increase in the channel width and the height of the vacuum caused by the capillary forces in the soil and a decrease in evaporation lead to an increase in the width of the zone of capillary spreading of the water l_k and to a broadening of the saturation zone. According to the data in the upper left and two lower blocks of Table 1, a change in the height of the capillary rise of the water, the capacity of the stratum and the width of the channel leads to an increase in the width L by 30, 113 and 118% respectively. However, evaporation from the free surface turns out to have the greatest effect on the width of the saturation zone: the data in the right-hand upper block of Table 1 show that, as the parameter ε decreases, the value of L increases by 144%.

Evaporation also turns out to have a significant effect on the width of the capillary spreading of the water. It follows from the same section of Table 1 that, as ε decreases, the magnitude of l_k increases by 190%. It is clear that the capillarity of the soil has the most significant effect on the parameter l_k: according to the data in the left-hand upper block of Table 1, the width of the capillary spread of the water increases by a factor of 10.8 when the parameter h_k becomes larger.

The lower right-hand block of Table 1 reflects a regularity which is natural from a physical point of view: an increase in the width of the saturation zone and, conversely, a decrease in the width of the capillary spread of the water, is accompanied by an increase in the capacity of the stratum T. So, as T changes, the width L increases by a factor of 2.13 and the width l_k decreases by a factor of almost 2.45.

It can be seen from the two right-hand blocks of Table 1 that, when $\varepsilon=0.4$ and $T=0.5$ and when $\varepsilon=0.1$ and $T=2$, we have l_k=0.4225 and l_k=0.4998 respectively and, consequently, the ratio l_k/h_k is equal to 1.0613 and 1.2495. Moreover, the values of the ratios become even higher when T and ε decrease. The significant effect of the horizontal intake of water (including in the case of weakly capillary soils), noted for the first time by Verigin, [15] is therefore confirmed.

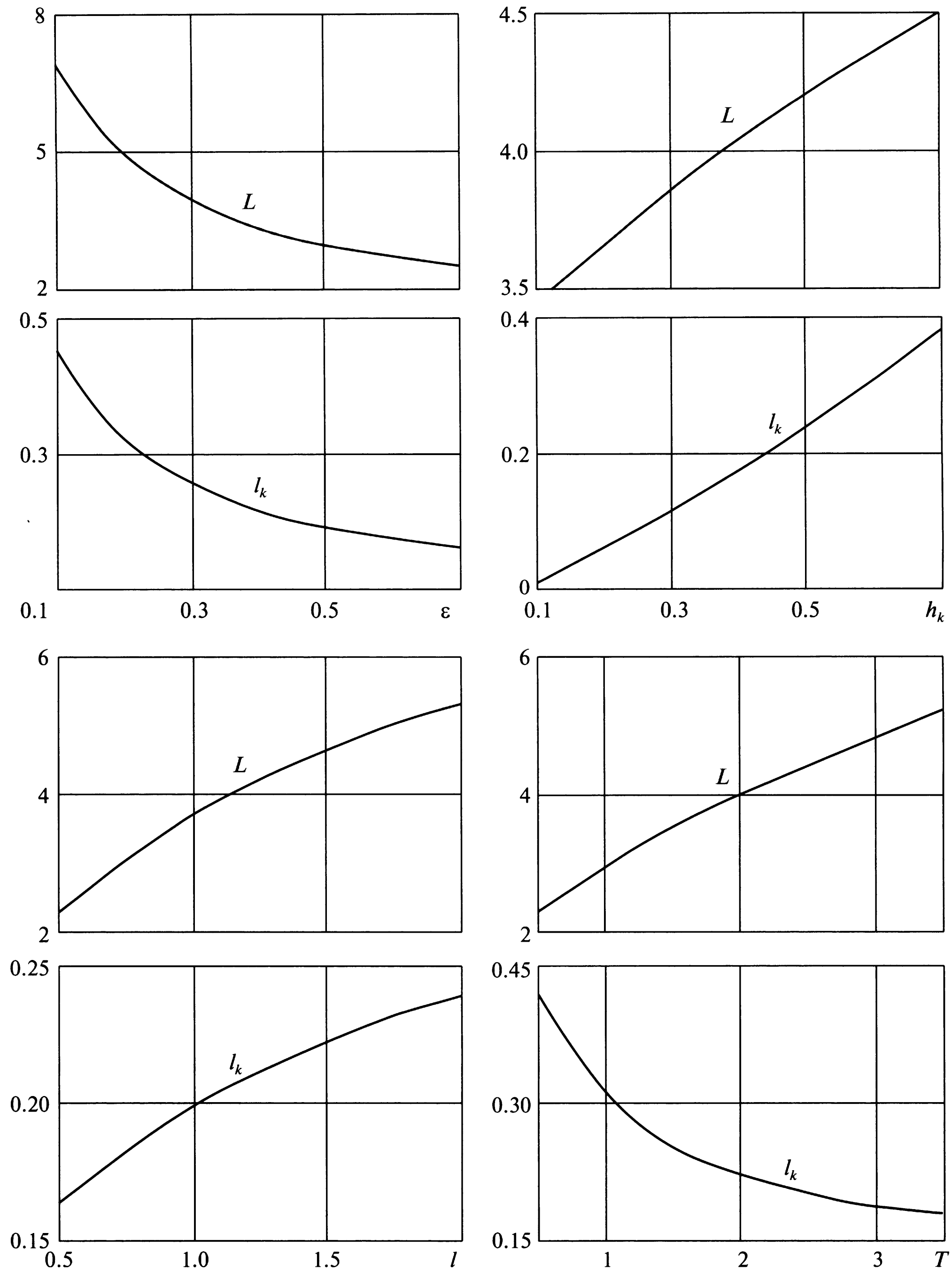

Fig. 5.

The nature of the effect of the governing physical parameters on the seepage flow rate can be judged from the right-hand section of all the blocks in Table 1: it is clear that the flow rate increases when all the parameters ε, h_k, l and T increase. At the same time, the flow rate depends significantly on the stratum capacity; it follows from the lower right-hand block of Table 1 that the flow rate can increase by a factor of 2.8.

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Translated by E. L. S.